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On Infinite-Dimensional Algebras of Symmetries of the Self-Dual Yang-Mills Equations

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Abstract

Infinite-dimensional algebras of hidden symmetries of the self-dual Yang-Mills equations are considered. A current-type algebra of symmetries and an affine extension of conformal symmetries introduced recently are discussed using the twistor picture. It is shown that the extended conformal symmetries of the self-dual Yang-Mills equations have a simple description in terms of Ward's twistor construction.

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I. INTRODUCTION

The self-dual Yang-Mills (SDYM) equations in the space R^4 are manifestly invariant under the group of gauge transformations and the group of conformal transformations of the space R^4. Besides these symmetries, the SDYM equations have infinite-dimensional algebras of 'hidden symmetries'[1–6] related to gauge transformations. In Refs. 1–5 the Yang gauge[7] in which two components of gauge potential are equal to zero was used. Crane generalized the results of Refs. 1–5 on the case when a gauge is not fixed and gave the twistor interpretation of the action of symmetries on solutions of the SDYM equations.[8]

In the frames of approach of Refs. 1–4 it has been shown[9] that on the space $\mathcal{M}$ of local solutions of the SDYM equations one can also define an (infinitesimal) action of the affine Lie algebra related to conformal transformations of R^4. Later, the action of the subalgebra of this algebra was also defined on the space of solutions of the vacuum self-dual gravity equations.[10] In this paper we shall show that the new symmetry algebra described in Ref. 9 has a simple interpretation in terms of vector fields acting on transition matrices of Ward's holomorphic vector bundles over twistor space.

II. DEFINITIONS AND NOTATION

We consider the Euclidean space $R^{4,0}$ with the metric $\delta_{\mu\nu}$ and potentials A_μ of the Yang-Mills (YM) fields $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu]$, where $\mu, \nu, ... = 1, ..., 4$. For simplicity we consider the fields A_μ and $F_{\mu\nu}$ with values in the Lie algebra $su(n)$.

The SDYM equations have the form

$$\frac{1}{2}\varepsilon_{\mu\nu\rho\sigma}F_{\rho\sigma} = F_{\mu\nu}, \tag{1}$$

where $\varepsilon_{\mu\nu\rho\sigma}$ is the completely antisymmetric tensor on $R^{4,0}$ and $\varepsilon_{1234} = 1$.

It is easily seen that eqs.(1) are invariant under the algebra of infinitesimal local gauge transformations:

$$\delta_\varphi A_\mu = \partial_\mu \varphi + [A_\mu, \varphi], \tag{2}$$

where $\varphi(x) \in su(n)$, $x \in R^{4,0}$. It is also well-known that the SDYM equations (1) are invariant with respect to the algebra $so(5,1)$ of infinitesimal conformal transformations:

$$\delta_N A_\mu = N^\nu \partial_\nu A_\mu + A_\nu \partial_\mu N^\nu, \quad \partial_\mu := \frac{\partial}{\partial x^\mu}, \tag{3}$$

where $N = N^\nu \partial_\nu$ is any generator of the 15-parameter conformal group:

$$X_a = \delta_{ab}\eta^b_{\mu\nu}x_\mu\partial_\nu,\ Y_a = \delta_{ab}\bar{\eta}^b_{\mu\nu}x_\mu\partial_\nu,\ P_\mu = \partial_\mu,$$

$$K_\mu = \frac{1}{2}x_\sigma x_\sigma \partial_\mu - x_\mu D,\ D = x_\sigma\partial_\sigma,\ a, b, ... = 1, 2, 3, \tag{4}$$

where $\{X_a\}$ and $\{Y_a\}$ generate two commuting $SO(3)$ subgroups in $SO(4)$, P_μ are the translation generators, K_μ are the generators of special conformal transformations and D is the dilatation generator. Here $\eta^a_{\mu\nu} = \{\varepsilon^a_{bc}, \mu = b, \nu = c;\ \delta^a_\mu, \nu = 4;\ -\delta^a_\nu, \mu = 4\}$ is the self-dual 't Hooft tensor and $\bar{\eta}^a_{\mu\nu} = \{\varepsilon^a_{bc}, \mu = b, \nu = c;\ -\delta^a_\mu, \nu = 4;\ \delta^a_\nu, \mu = 4\}$ is the anti-self-dual 't Hooft tensor.

Let $J = (J_\mu^\nu)$ be the most general complex structure on $R^{4,0}$:[11]

$$J_\mu^\nu = s_a \bar{\eta}^a_{\mu\sigma}\delta^{\sigma\nu} \implies J_\mu^\sigma J_\sigma^\nu = -\delta_\mu^\nu, \tag{5}$$

where real numbers s_a parametrize a two-sphere S^2, $s_a s_a = 1$. Using J, one can introduce $(0,1)$ vector fields

$$\bar{V}_{\bar{1}} = \frac{1}{2}(\partial_1 + i\partial_2) - \frac{\lambda}{2}(\partial_3 + i\partial_4) = \partial_{\bar{y}} - \lambda\partial_z,\ \bar{V}_{\bar{2}} = \frac{1}{2}(\partial_3 - i\partial_4) + \frac{\lambda}{2}(\partial_1 - i\partial_2) = \partial_{\bar{z}} + \lambda\partial_y, \tag{6}$$

where $y = x_1 + ix_2, z = x_3 - ix_4, \bar{y} = x_1 - ix_2, \bar{z} = x_3 + ix_4$ are complex coordinates on $R^4 \simeq C^2$, and $\lambda = (s_1 + is_2)/(1 + s_3)$ is a local complex coordinate on $S^2 \simeq CP^1$.

Let S^1 denote a contour $|\lambda| = 1$ in the λ-plane enclosing the origin in the counterclockwise direction, C_+ denote all complex numbers $\lambda \in CP^1$ with $|\lambda| \le 1+\alpha$, where α is any positive real number, and C_- denote all complex numbers $\lambda \in CP^1$ with $|\lambda| \ge 1-\alpha$ (including $\lambda = \infty$). Then C_+ and C_- form a two-set open cover of the Riemann sphere CP^1 with the intersection $C_\alpha = C_+ \cap C_- = \{\lambda :\ 1-\alpha \le |\lambda| \le 1+\alpha\}$. Vector field $\partial_{\bar{\lambda}} := \partial/\partial_{\bar{\lambda}}$ is antiholomorphic $(0,1)$ vector field with respect to the standard complex structure $\varepsilon = i\, d\lambda \otimes \partial_\lambda - i\, d\bar{\lambda} \otimes \partial_{\bar{\lambda}}$ on S^2.

Twistor space $\mathcal{Z}$ for $R^{4,0}$ is the space $\mathcal{Z} = R^{4,0} \times CP^1 \simeq CP^3 - CP^1 \subset CP^3$ which is the space of all the constant complex structures on $R^{4,0}$.[12] This space can be covered by two coordinate patches $\mathcal{Z} = U_+ \cup U_-$:

$$U_+ = \{x \in R^{4,0}, \lambda \in C_+\}, \quad U_- = \{x \in R^{4,0}, \lambda \in C_-\}, \tag{7}$$

with the intersection $U = U_+ \cap U_- = \{x \in R^{4,0}, \lambda \in C_\alpha = C_+ \cap C_-\}$.

Twistor space $\mathcal{Z}$ is a trivial bundle $\pi :\ \mathcal{Z} \to R^{4,0}$ over $R^{4,0}$ ($\pi :\ \{x_\mu, \lambda, \bar{\lambda}\} \to \{x_\mu\}$) with fibre CP^1 and is a complex manifold with complex structure $\mathcal{J} = (J, \varepsilon)$ on $\mathcal{Z}$. Vector fields $\bar{V}_{\bar{1}}, \bar{V}_{\bar{2}}$ from (6) and $\bar{V}_{\bar{3}} = \partial_{\bar{\lambda}}$ are vector fields of type $(0,1)$ with respect to complex structure $\mathcal{J}$. Holomorphic coordinates on U_+ and U_- are $\{v_+^1 = y - \lambda\bar{z}, v_+^2 = z + \lambda\bar{y}, v_+^3 = \lambda\}$ and $\{v_-^1 = \frac{1}{\lambda}y - \bar{z}, v_-^2 = \frac{1}{\lambda}z + \bar{y}, v_-^3 = \frac{1}{\lambda}\}$. Twistor space is also a nontrivial holomorphic bundle $p :\ \mathcal{Z} \to CP^1$ over CP^1 with holomorphic sections (projective lines CP^1_x) parametrized by points $x \in C^4$:

$$CP^1_x = \{v_+^1 = y + \lambda\tilde{z},\ v_+^2 = z + \lambda\tilde{y},\ (y, z, \tilde{y}, \tilde{z}) \in C^4\}. \tag{8}$$

On $\mathcal{Z}$ one can introduce a real structure as a map $\tau :\ R^{4,0} \times S^2 \to R^{4,0} \times S^2$ defined by the formula: $\tau(x_\mu, \lambda) = (x_\mu, -1/\bar{\lambda}),\ x_\mu \in R^{4,0}$. Then, for real holomorphic sections of the bundle $\mathcal{Z} \to CP^1$ satisfying the equations $\tau(v_+^1) = \overline{v_-^2},\ \tau(v_+^2) = -\overline{v_-^1}$ we have $\tilde{z} = -\bar{z},\ \tilde{y} = \bar{y}$.[12,11]

III. WARD'S TWISTOR CORRESPONDENCE AND GAUGE-TYPE HIDDEN SYMMETRIES

The potentials A_μ define a connection $D := dx^\mu(\partial_\mu + A_\mu)$ in the principal $SU(n)$-bundle $P = P(R^{4,0}, SU(n))$ over $R^{4,0}$. In the standard manner we introduce the complex vector bundle $E = P \times_{SU(n)} C^n$ associated to P. Sections of this bundle are C^n-valued vector-functions depending on $x \in R^{4,0}$. Using the projection $\pi :\ \mathcal{Z} \to R^{4,0}$, we can pull back the bundle E with self-dual connection D to the bundle $\tilde{E} := \pi^* E$ over $\mathcal{Z}$,

and the pulled back connection $\tilde{D} := \pi^* D$ will be flat along the fibres CP^1_x of the bundle $\mathcal{Z} \to R^{4,0}$. We can take it in the form $\tilde{D} = D + d\lambda\partial_\lambda + d\bar{\lambda}\partial_{\bar{\lambda}}$.

Let us consider local holomorphic sections ξ of the bundle $\tilde{E}$ or, in other words, local solutions of the equations $\tilde{D}^{(0,1)}_{\bar{a}}\xi = 0$:

$$[(D_1 + iD_2) - \lambda(D_3 + iD_4)]\xi(x, \lambda, \bar{\lambda}) = 0, \tag{9a}$$

$$[(D_3 - iD_4) + \lambda(D_1 - iD_2)]\xi(x, \lambda, \bar{\lambda}) = 0, \tag{9b}$$

$$\partial_{\bar{\lambda}}\xi(x, \lambda, \bar{\lambda}) = 0, \tag{10}$$

where $\tilde{D}^{(0,1)}_{\bar{a}}$ are components of $\tilde{D}$ along $(0, 1)$ vector fields $\bar{V}_{\bar{a}}$. We can solve eq.(10), and then eqs.(9) on $\xi(x, \lambda)$ are usually called the linear system for the SDYM equations.[13,14] It is easily seen that the compatibility conditions of the linear system (9) coincide with the SDYM equations (1).

Equations (9) have a local solution $\xi_+(x, \lambda)$ over $U_+ \subset \mathcal{Z}$, a local solution $\xi_-(x, \lambda)$ over $U_- \subset \mathcal{Z}$ and $\xi_+ = \xi_-$ on $U_+ \cap U_-$. We can always represent $\xi_\pm$ in the form $\xi_\pm = \psi_\pm\chi_\pm$, where $SL(n, C)$-valued functions $\psi_\pm$ are holomorphic in λ on $U_\pm$, and vector-functions $\chi_\pm \in C^n$ defined on $U_\pm$ are related by

$$\chi_- = \mathcal{F}\chi_+ \tag{11}$$

on $U = U_+ \cap U_- \subset \mathcal{Z}$. Here $\chi_+ = \chi_+(v^a_+)$ and $\chi_- = \chi_-(v^a_-)$ are Čech fibre coordinates of the bundle $\tilde{E}$ over U_+ and U_-, and $\mathcal{F}$ is the transition matrix in the bundle $\tilde{E}$. Matrix $\mathcal{F}$ is a holomorphic $SL(n, C)$-valued function on U with non-vanishing determinant. By Ward's construction, the bundle $\tilde{E}$ is holomorphically trivial on each fibre CP^1_x, i.e. $\tilde{E}\mid_{CP^1_x} = CP^1_x \times C^n$.[14] This means that on CP^1_x matrix $\mathcal{F}$ can be factorized in the form

$$\mathcal{F} = \psi_-^{-1}(x, \lambda)\psi_+(x, \lambda) \tag{12}$$

for each point x from an open subset of $R^{4,0}$.

Remark. Let we are given a matrix-valued function $\mathcal{F} \in G$ on $C_\alpha = C_+ \cap C_- \subset CP^1$. Then Riemann-Hilbert problem is to find matrix-valued functions $\psi_+ \in G$ and $\psi_- \in G$ such that ψ_+ is regular (i.e. holomorphic with non-vanishing determinant) on C_+, ψ_- is regular on C_- and $\mathcal{F} = \psi_-^{-1}\psi_+$ is regular on $C_\alpha = C_+ \cap C_-$. Solution of the Riemann-Hilbert problem exists (Birkhoff's theorem) and is unique up to the left multiplication by matrix g not depending on λ : $\psi_\pm \to g\psi_\pm$ (for discussion see, e.g., Ref. 15). Thus, the splitting (12) gives a solution of a parametric Riemann-Hilbert problem (x_μ are external parameters). Besides, the matrix $\mathcal{F}$ in (12) will not change if we replace $\psi_\pm(x, \lambda)$ by $g(x)\psi_\pm(x, \lambda)$ (standard gauge transformation).

Matrix-valued functions $\psi_\pm$ define a trivialization of the bundle $\tilde{E}$ over $U_\pm$. It follows from (9)–(12) that

$$(\partial_{\bar{y}}\psi_+ - \lambda\partial_z\psi_+)\psi_+^{-1} = (\partial_{\bar{y}}\psi_- - \lambda\partial_z\psi_-)\psi_-^{-1} = -(A_{\bar{y}}(x) - \lambda A_z(x)), \tag{13a}$$

$$(\partial_{\bar{z}}\psi_+ + \lambda\partial_y\psi_+)\psi_+^{-1} = (\partial_{\bar{z}}\psi_- + \lambda\partial_y\psi_-)\psi_-^{-1} = -(A_{\bar{z}}(x) + \lambda A_y(x)), \tag{13b}$$

and the potentials $\{A_\mu\}$ defined by (13) satisfy the SDYM equations.

For a Hermitian vector bundle when the structure group is $SU(n)$ the real structure τ on the twistor space $\mathcal{Z}$ induces a Hermitian structure in the bundle $\tilde{E}$.[12] Then, fields $\{A_\mu\}$, matrices $\psi_\pm$ and $\mathcal{F}$ have to satisfy the following unitarity conditions (see e.g. Ref. 8):

$$A_y^\dagger = -A_{\bar{y}},\ A_z^\dagger = -A_{\bar{z}},\ A_{\bar{y}}^\dagger = -A_y,\ A_{\bar{z}}^\dagger = -A_z, \tag{14a}$$

$$\psi_+^\dagger(\lambda) = \psi_-^{-1}(-\frac{1}{\bar\lambda}),\ \psi_-^\dagger(\lambda) = \psi_+^{-1}(-\frac{1}{\bar\lambda}), \tag{14b}$$

$$\mathcal{F}^\dagger(\lambda) = \mathcal{F}(-\frac{1}{\bar\lambda}), \tag{14c}$$

where † denotes Hermitian conjugation.

From (13) one can see that gauge potentials A_μ do not change after transformations

$$\psi_+ \to \psi_+^{eqv} = \psi_+ h_+,\ \psi_- \to \psi_-^{eqv} = \psi_- h_-, \tag{15}$$

where h_+ is any regular holomorphic matrix-valued function on U_+, and h_- is any regular holomorphic matrix-valued function on U_-. This means that the bundles with transition matrices $h_-^{-1}\mathcal{F}h_+$ and $\mathcal{F}$ are holomorphically equivalent. Notice that for the Hermitian vector bundles $\tilde E$ with transition matrices $\mathcal{F}$ satisfying the 'hermiticity' condition (14c) the equality $h_+^\dagger(\lambda) = h_-^{-1}(-1/\bar\lambda)$ has to be fulfilled.

Thus, we have described a one-to-one correspondence between gauge equivalence classes of solutions to the SDYM equations on the Euclidean 4-space and equivalence classes of holomorphic vector bundles $\tilde E$ over twistor space $\mathcal{Z}$, that are holomorphically trivial over each real projective line CP_x^1 in $\mathcal{Z}$ (the Euclidean version of Ward's theorem[16,12,11]).

Now we will briefly remind the description of gauge-type 'hidden symmetries'. First, we should define an infinite-dimensional complex Lie (pseudo)group H (current-type group) of holomorphic maps from $U = U_+ \cap U_- \subset \mathcal{Z}$ to the group $SL(n, C)$ with standard pointwise multiplication and the corresponding complex Lie algebra $\mathcal{H}$ of holomorphic maps from U to the Lie algebra $sl(n, C)$. We shall also consider subgroups $H_\pm \subset H$ of elements from H which can be extended continuously to holomorphic maps from $U_\pm$ to $SL(n, C)$ and the corresponding algebras $\mathcal{H}_\pm \subset \mathcal{H}$.

The action of group H on transition matrix $\mathcal{F}$ preserving the hermiticity condition (14c) is given by:[8]

$$\mathcal{F} \to \tilde{\mathcal{F}} = u(h)\mathcal{F} := h(\lambda)\mathcal{F}(\lambda)h^\dagger(-\frac{1}{\bar\lambda}), \tag{16}$$

where

$$h(\lambda) \equiv h(y - \lambda\bar z, z + \lambda\bar y, \lambda), \mathcal{F} \equiv \mathcal{F}(y - \lambda\bar z, z + \lambda\bar y, \lambda), h^\dagger(-\frac{1}{\bar\lambda}) \equiv h^\dagger(y + \frac{1}{\bar\lambda}\bar z, z - \frac{1}{\bar\lambda}\bar y, -\frac{1}{\bar\lambda}),$$

and in (16) we simply omitted a part of arguments. To this action the following action of the Lie algebra $\mathcal{H}$ on $\mathcal{F}$ corresponds:

$$\mathcal{F} \to \delta_\varphi\mathcal{F} = \varphi(\lambda)\mathcal{F}(\lambda) + \mathcal{F}(\lambda)\varphi^\dagger(-\frac{1}{\bar\lambda}), \tag{17}$$

where $\varphi(\lambda) \equiv \varphi(y - \lambda\bar z, z + \lambda\bar y, \lambda) \in sl(n, C)$ is any element of $\mathcal{H}$ (of course, $\varphi^\dagger(-\frac{1}{\bar\lambda})$ is also an element of $\mathcal{H}$).

Consider now the following $sl(n, C)$-valued function ϕ on U

$$\phi \equiv \varphi_- - \varphi_+ := \psi_-(\delta_\varphi\mathcal{F})\psi_+^{-1} = \psi_-\varphi(\lambda)\psi_-^{-1} + \psi_+\varphi^\dagger(-\frac{1}{\bar\lambda})\psi_+^{-1}, \tag{18}$$

where for holomorphic ϕ expanded in Laurent series $\phi = \sum_{n=-\infty}^{\infty}\lambda^n\phi_n(x)$ we put

$$\varphi_+ := \tilde\phi_0(x) - \sum_{n=1}^{\infty}\lambda^n\phi_n(x),\ \varphi_- := \hat\phi_0(x) + \sum_{n=-\infty}^{-1}\lambda^n\phi_n(x),\ \hat\phi_0(x) - \tilde\phi_0(x) = \phi_0(x). \tag{19}$$

Thus, the function $\varphi_+(\lambda) \in sl(n,C)$ is holomorphic in $\lambda \in C_+ \subset CP^1$, and the function $\varphi_-(\lambda) \in sl(n,C)$ is holomorphic in $\lambda \in C_- \subset CP^1$.

It follows from (13) and (18) that

$$(D_{\bar y}-\lambda D_z)\phi \equiv (\partial_{\bar y}-\lambda\partial_z)\phi+[A_{\bar y}-\lambda A_z,\phi] = 0 \implies (D_{\bar y}-\lambda D_z)\varphi_+ = (D_{\bar y}-\lambda D_z)\varphi_-, \quad (20a)$$

$$(D_{\bar z}+\lambda D_y)\phi \equiv (\partial_{\bar z}+\lambda\partial_y)\phi+[A_{\bar z}+\lambda A_y,\phi] = 0 \implies (D_{\bar z}+\lambda D_y)\varphi_+ = (D_{\bar z}+\lambda D_y)\varphi_-. \quad (20b)$$

The action of algebra $\mathcal{H}$ on matrix-valued functions $\psi_\pm \in SL(n,C)$ and on gauge potentials $\{A_\mu\}$ is given by formulae

$$\delta_\varphi\psi_+ := -\varphi_+\psi_+,\ \delta_\varphi\psi_- := -\varphi_-\psi_-, \tag{21}$$

$$\delta_\varphi A_{\bar y} - \lambda\delta_\varphi A_z := D_{\bar y}\varphi_+ - \lambda D_z\varphi_+ = D_{\bar y}\varphi_- - \lambda D_z\varphi_-, \quad (22a)$$

$$\delta_\varphi A_{\bar z} + \lambda\delta_\varphi A_y := D_{\bar z}\varphi_+ + \lambda D_y\varphi_+ = D_{\bar z}\varphi_- + \lambda D_y\varphi_-. \quad (22b)$$

It follows from (22) that

$$\delta_\varphi A_y = \oint_{S^1}\frac{d\lambda}{2\pi i\lambda^2}(D_{\bar z}\varphi_+ + \lambda D_y\varphi_+),\ \delta_\varphi A_z = -\oint_{S^1}\frac{d\lambda}{2\pi i\lambda^2}(D_{\bar y}\varphi_+ - \lambda D_z\varphi_+),$$

$$\delta_\varphi A_{\bar y} = \oint_{S^1}\frac{d\lambda}{2\pi i\lambda}(D_{\bar y}\varphi_+ - \lambda D_z\varphi_+),\ \delta_\varphi A_{\bar z} = \oint_{S^1}\frac{d\lambda}{2\pi i\lambda}(D_{\bar z}\varphi_+ + \lambda D_y\varphi_+), \tag{23}$$

where $S^1 = \{\lambda \in CP^1 : |\lambda| = 1\}$. Thus, in (17), (21) and (22) we have described the action of the current algebra $\mathcal{H}$ on the space of local solutions of the SDYM equations.

IV. HIDDEN SYMMETRIES FROM THE AFFINIZATION OF CONFORMAL ALGEBRA

In Ref. 9 to each generator N of the group $SO(5,1)$ of conformal transformations of the Euclidean space $R^{4,0}$ Popov and Preitschopf associated an infinite number of infinitesimal symmetry transformations δ_N^n ($n = 0,1,2,...$) acting on solutions of the SDYM equations. They considered potentials A_μ with values in complex Lie algebras and did not discuss the problems of reality for gauge fields. Considering potentials A_μ with values in $su(n)$, we examine not only the infinitesimal symmetries from Ref. 9, which do not preserve, generally speaking, the unitarity conditions (14), but also the symmetries preserving the unitarity.

In formulae (4) we have specified a homomorphism of the Lie algebra $so(5,1)$ of the conformal group into the Lie algebra of vector fields $M,N,...$ on $R^{4,0}$. Now we have to define the action of $SO(5,1)$ on $\mathcal{Z}$, which preserves the holomorphicity of the bundle $\tilde E \to \mathcal{Z}$. Such lift $N \to \tilde N$ of vector fields on $\mathcal{Z}$ was described in Ref. 17, and the lifted generators

$$\tilde X_a = X_a,\ \tilde Y_a = Y_a + 2Z_a,\ \tilde P_\mu = P_\mu,\ \tilde K_\mu = K_\mu + \bar\eta^a_{\sigma\mu}x_\sigma Z_a,\ \tilde D = D, \tag{24}$$

$$Z_1 = \frac{1}{2}i(\lambda^2-1)\partial_\lambda - \frac{1}{2}i(\bar\lambda^2-1)\partial_{\bar\lambda},\ Z_2 = \frac{1}{2}(\lambda^2+1)\partial_\lambda + \frac{1}{2}(\bar\lambda^2+1)\partial_{\bar\lambda},\ Z_3 = i\lambda\partial_\lambda - i\bar\lambda\partial_{\bar\lambda} \tag{25}$$

are infinitesimal automorphisms of the complex structure $\mathcal{J}$ on $\mathcal{Z}$. This means, in particular, that for any generator $\tilde N$ from (24) the following relations

$$[\bar V_{\bar a}, \tilde N] = \alpha_{\bar a}^{\bar b}(\tilde N)\bar V_{\bar b} \tag{26}$$

take place for some functions $\alpha_{\bar{a}}^{\bar{b}}(\tilde{N})$ and vector fields $\bar{V}_{\bar{1}}, \bar{V}_{\bar{2}}$ from (6) and $\bar{V}_{\bar{3}} = \partial_{\bar{\lambda}}$.

Having generators $\tilde{N}$ of infinitesimal holomorphic diffeomorphisms (24) of the space $\mathcal{Z}$, we consider the holomorphic vector fields $N_n = \lambda^{-n}\tilde{N},\ n = 0, \pm 1, \pm 2, ...$, and define the following infinitesimal transformations of transition matrix $\mathcal{F}$ of the bundle $\tilde{E}$

$$\delta_N^n \mathcal{F} := N_n(\mathcal{F}) = \lambda^{-n}\tilde{N}(\mathcal{F}),\ n = 0, \pm 1, \pm 2, ... \ . \tag{27}$$

In virtue of (26) transformations (27) are holomorphic and, therefore, define the infinitesimal symmetries of the SDYM equations.

Let us introduce the $sl(n,C)$-valued function (compare with (18) from Sec. III)

$$\phi_{N_n} = \phi_{N_n}^- - \phi_{N_n}^+ := \psi_-(\delta_N^n \mathcal{F})\psi_+^{-1} = \lambda^{-n}(\tilde{N}\psi_+)\psi_+^{-1} - \lambda^{-n}(\tilde{N}\psi_-)\psi_-^{-1}, \tag{28}$$

$$\phi_{N_n}^+ := \tilde{\phi}_{N_n}^0(x) - \sum_{k=1}^{\infty}\lambda^k \phi_{N_n}^k(x),\ \phi_{N_n}^- := \hat{\phi}_{N_n}^0(x) + \sum_{k=-\infty}^{-1}\lambda^k \phi_{N_n}^k(x),\ \hat{\phi}_{N_n}^0 - \tilde{\phi}_{N_n}^0 = \phi_{N_n}^0. \tag{29}$$

Remind that the splitting of the $sl(n,C)$-valued function $\phi_{N_n}(x,\lambda)$ into the difference of the function $\phi_{N_n}^- \in sl(n,C)$ holomorphic in $\lambda \in C_-$ and of the function $\phi_{N_n}^+ \in sl(n,C)$ holomorphic in $\lambda \in C_+$ is a solution of the infinitesimal variant of Riemann-Hilbert problem.

Remark. Notice that $\phi_{N_0}^\pm = -(\tilde{N}\psi_\pm)\psi_\pm^{-1}$, but $\phi_{N_n}^+ \neq -\lambda^{-n}(\tilde{N}\psi_+)\psi_+^{-1}$, and $\phi_{N_n}^- \neq -\lambda^{-n}(\tilde{N}\psi_-)\psi_-^{-1}$ when $n \neq 0$. If we fix the gauge $\psi_+(\lambda = 0) = 1$, then the functions $\phi_{N_0}^+$ coincide with the functions $-\psi_{\tilde{N}}, \tilde{N} \in so(5,1)$, which are generating functions for symmetries introduced in Ref. 9.

It follows from (13), (26) and (28) that (cf. Sec. III):

$$(D_{\bar{y}} - \lambda D_z)\phi_{N_n} = 0 \implies (D_{\bar{y}} - \lambda D_z)\phi_{N_n}^+ = (D_{\bar{y}} - \lambda D_z)\phi_{N_n}^-, \tag{30a}$$

$$(D_{\bar{z}} + \lambda D_y)\phi_{N_n} = 0 \implies (D_{\bar{z}} + \lambda D_y)\phi_{N_n}^+ = (D_{\bar{z}} + \lambda D_y)\phi_{N_n}^-, \tag{30b}$$

$$\delta_N^n \psi_+ = -\phi_{N_n}^+\psi_+,\ \delta_N^n \psi_- = -\phi_{N_n}^-\psi_-, \tag{31}$$

$$\begin{aligned} \delta_N^n A_{\bar{y}} - \lambda\delta_N^n A_z &:= D_{\bar{y}}\phi_{N_n}^+ - \lambda D_z\phi_{N_n}^+ = D_{\bar{y}}\phi_{N_n}^- - \lambda D_z\phi_{N_n}^-, \\ \delta_N^n A_{\bar{z}} + \lambda\delta_N^n A_y &:= D_{\bar{z}}\phi_{N_n}^+ + \lambda D_y\phi_{N_n}^+ = D_{\bar{z}}\phi_{N_n}^- + \lambda D_y\phi_{N_n}^-. \end{aligned} \tag{32}$$

We have

$$\delta_N^n A_y = \oint_{S^1}\frac{d\lambda}{2\pi i\lambda^2}(D_{\bar{z}}\phi_{N_n}^+ + \lambda D_y\phi_{N_n}^+),\ \delta_N^n A_z = -\oint_{S^1}\frac{d\lambda}{2\pi i\lambda^2}(D_{\bar{y}}\phi_{N_n}^+ - \lambda D_z\phi_{N_n}^+),$$

$$\delta_N^n A_{\bar{y}} = \oint_{S^1}\frac{d\lambda}{2\pi i\lambda}(D_{\bar{y}}\phi_{N_n}^+ - \lambda D_z\phi_{N_n}^+),\ \delta_N^n A_{\bar{z}} = \oint_{S^1}\frac{d\lambda}{2\pi i\lambda}(D_{\bar{z}}\phi_{N_n}^+ + \lambda D_y\phi_{N_n}^+), \tag{33}$$

Thus, to each generator N of the group of conformal transformations of the space $R^{4,0}$ we have corresponded an infinite number of generators δ_N^n of symmetry transformations of the SDYM equations and defined their action on the transition matrix $\mathcal{F}$, on the group-valued functions $\psi_\pm$ and on gauge potentials A_μ.

By direct calculations, it is not hard to show that

$$\delta_N^0 A_\mu = \mathcal{L}_N A_\mu \equiv N^\nu\partial_\nu A_\mu + A_\nu\partial_\mu N^\nu, \tag{34}$$

i.e. $\delta_N^0 A_\mu$ coincide with infinitesimal conformal transformation (3). Symmetries (33) with $n \geq 0$ are in one-to-one correspondence with the symmetries introduced in Ref. 9. To

show this correspondence in more details, let us discuss the algebraic properties of the symmetry transformations δ^n_N.

The commutator of two transformations can be easily calculated using formula (27). We have

$$[\delta^m_M, \delta^n_N]\mathcal{F} = \lambda^{-m-n}[\tilde{M}, \tilde{N}](\mathcal{F}) + m\lambda^{-m-n-1}\tilde{N}^\lambda \tilde{M}(\mathcal{F}) - n\lambda^{-m-n-1}\tilde{M}^\lambda \tilde{N}(\mathcal{F}), \qquad (35)$$

where $\tilde{N}^\lambda$ and $\tilde{M}^\lambda$ are components of vector fields $\tilde{N}$ and $\tilde{M}$ along ∂_λ. Substituting (24) into (35), we obtain

$$[\delta^m_M, \delta^n_N] = \delta^{m+n}_{[M,N]}, \;\; m, n, = 0, \pm 1, \pm 2, ... \qquad (36)$$

for $M, N \in \mathcal{A} = \{P_\mu, X_a, D\}$, i.e. δ^m_M with generators M from the 8-dimensional Lie algebra $\mathcal{A}$ (see Ref. 9) form the affine Lie algebra $\mathcal{A} \otimes C[\lambda, \lambda^{-1}]$. For generators $\delta^m_{Y_a}$ we have

$$[\delta^m_{Y_1}, \delta^n_{Y_1}] = i(m-n)(\delta^{m+n-1}_{Y_1} - \delta^{m+n+1}_{Y_1}), \qquad (37a)$$

$$[\delta^m_{Y_2}, \delta^n_{Y_2}] = (m-n)(\delta^{m+n-1}_{Y_2} + \delta^{m+n+1}_{Y_2}), \qquad (37b)$$

$$[\delta^m_{Y_3}, \delta^n_{Y_3}] = 2i(m-n)\delta^{m+n}_{Y_3}, \qquad (37c)$$

$$[\delta^m_{Y_1}, \delta^n_{Y_2}] = \delta^{m+n}_{[Y_1,Y_2]} + m(\delta^{m+n-1}_{Y_1} + \delta^{m+n+1}_{Y_1}) - in(\delta^{m+n-1}_{Y_2} - \delta^{m+n+1}_{Y_2}), \qquad (38a)$$

$$[\delta^m_{Y_1}, \delta^n_{Y_3}] = \delta^{m+n}_{[Y_1,Y_3]} + 2im\delta^{m+n}_{Y_1} - in(\delta^{m+n-1}_{Y_3} - \delta^{m+n+1}_{Y_3}), \qquad (38b)$$

$$[\delta^m_{Y_2}, \delta^n_{Y_3}] = \delta^{m+n}_{[Y_2,Y_3]} + 2im\delta^{m+n}_{Y_2} - n(\delta^{m+n-1}_{Y_3} + \delta^{m+n+1}_{Y_3}), \qquad (38c)$$

$$[\delta^m_{Y_1}, \delta^n_N] = \delta^{m+n}_{[Y_1,N]} - in(\delta^{m+n-1}_N - \delta^{m+n+1}_N), \qquad (39a)$$

$$[\delta^m_{Y_2}, \delta^n_N] = \delta^{m+n}_{[Y_2,N]} - n(\delta^{m+n-1}_N + \delta^{m+n+1}_N), \qquad (39b)$$

$$[\delta^m_{Y_3}, \delta^n_N] = \delta^{m+n}_{[Y_3,N]} - 2in\delta^{m+n}_N, \qquad (39c)$$

where $N \in \mathcal{A} = \{P_\mu, X_a, D\}$. Notice that the imaginary unit i can be removed from structure constants in (37)-(39) by the following change of generators $\delta^m_{Y_1} \to i\delta^m_{Y_1}$, $\delta^m_{Y_3} \to i\delta^m_{Y_3}$ (see Ref. 9). Then "half" of the algebra (36)-(39) with $m, n \geq 0$ will coincide with the one considered in Ref. 9. Let us emphasize that in (36)-(39) the numbers $m, n, = 0, \pm 1, ...$ are any integer numbers. It is important to undestand that the change $\delta^m_{Y_1} \to i\delta^m_{Y_1}$, $\delta^m_{Y_3} \to i\delta^m_{Y_3}$ is possible only by considering complex gauge fields or by transition from the signature $(4, 0)$ to the signature $(2, 2)$.

Finally, for generators K_μ of special conformal transformations, the commutators of transformations $\delta^m_{K_\mu}$ with other transformations will be again the symmetry transformations, but with coefficients depending on the coordinates on $\mathcal{Z}$. For example,

$$[\delta^m_{P_1}, \delta^n_{K_1}] = \delta^{m+n}_{[P_1,K_1]} + \frac{m}{2}v^2_+\delta^{m+n+1}_{P_1} - \frac{m}{2}v^1_+\delta^{m+n}_{P_1}, \qquad (40)$$

where $v^1_+ = y - \lambda\bar{z}, v^2_+ = z + \lambda\bar{y}$ are holomorphic coordinates on $U_+ \subset \mathcal{Z}$. This means that the structure constants in all the commutators with $\delta^m_{K_\mu}$ are replaced by the "structure functions", that has been noticed by Popov and Preitschopf.[9] We do not write out these commutators looking similar to (40). In twistor terms, commutators like (40) appear because of the fact that the expansion in Laurent series in λ and formulae like (27), (29) etc. may be used only by the existence of the holomorphic projection $p : \mathcal{Z} \to CP^1$ leading to the distinguishing of coordinate λ. Special conformal transformations K_μ and

transformations $\delta^m_{K_\mu}$ related to K_μ do not preserve the bundle $\mathcal{Z} \to CP^1$ and make invalid the formulae based on the expansion in λ. On formal level, this is connected with the fact that special conformal transformations transform λ into a function on x_μ and λ, and this leads to appearence of "structure functions" in formulae like (40). For correct discussion of symmetries related to K_μ, it is necessary to pass from the flat hyperKähler space $R^{4,0}$ to the four-sphere S^4 which is conformally flat (but not hyperKähler) manifold.

Remind that for real gauge fields with values in the Lie algebra $su(n)$ the transition functions $\mathcal{F}$ in the bundle $\tilde{E}$ have to satisfy the condition (14c). Generally speaking, infinitesimal transformations do not (infinitesimally) preserve this condition, since

$$(\delta^n_N \mathcal{F}\mid_{\lambda\to-\frac{1}{\lambda}})^\dagger = (-1)^n\lambda^n\tilde{N}\mathcal{F}(\lambda) = (-1)^n\delta^{-n}_N\mathcal{F}(\lambda). \tag{41}$$

However, we can introduce infinitesimal symmetry transformations $\delta^n_{N_R}$ preserving reality conditions by setting

$$\delta^n_{N_R}\mathcal{F} := \delta^n_N\mathcal{F} + (-1)^n\delta^{-n}_N\mathcal{F} = (\lambda^{-n} + (-\lambda)^n)\tilde{N}\mathcal{F} \equiv N^R_n\mathcal{F}. \tag{42}$$

For transformations (42), the formulae for $\phi_{N^R_n}$, $\delta^n_{N_R}\psi_\pm$ and $\delta^n_{N_R}A_\mu$ will have the same form as the formulae (28)-(33) with the replacement N_n by N^R_n, and that is why we shall not write out them. As to the algebraic properties of transformations (42), then, by direct calculation, it can be shown that generators $\delta^n_{P^R_\mu}$, $\delta^n_{X^R_a}$ and $\delta^n_{D^R}$ form a closed algebra. Commutation relations analogous to (36)-(39), which we do not write out, easily follow from the definition (42).

V. CONCLUSION

Symmetries of the SDYM and the self-dual gravity equations are important in quantization of Yang-Mills model, N=2 strings and related models, that has been discussed, for example, in recent papers.[18,19] We think that the twistor point of view on symmetries of the SDYM equations used in this paper clarifies the geometric and algebraic sence of the so called 'hidden symmetries' and makes them not so mystical. It would be interesting to consider the affine Lorentz symmetries[10] of the self-dual gravity equations from the twistor point of view. Notice that all the results of our paper may be generalized on the SDYM equations in the $4k$-dimensional spaces considered e.g. in Ref. 20.

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